RECOVERING SIGNALS FROM THE SHORT-TIME FOURIER TRANSFORM MAGNITUDE

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ABSTRACT

The problem of recovering signals from the Short-Time Fourier Transform (STFT) magnitude is of paramount importance in many areas of engineering and physics. This problem has received a lot of attention over the last few decades, but not much is known about conditions under which the STFT magnitude is a unique signal representation. Also, the recovery techniques proposed by researchers are mostly heuristic in nature. In this work, we first show that almost all signals can be uniquely identified by their STFT magnitude under mild conditions. Then, we consider a semidefinite relaxation-based algorithm and provide the first theoretical guarantees for the same. Numerical simulations complement our theoretical analysis and provide many directions for future work.

Index Terms— Short-Time Fourier Transform magnitude, unique signal representation, semidefinite relaxation.

1. INTRODUCTION

Signal recovery from the magnitude of the Fourier transform is known as phase retrieval. This recovery problem occurs in many fields, such as X-ray crystallography [1], astronomical imaging [2], speech recognition [3], computational biology [4] and blind channel estimation [5]. A considerable amount of work has been done by researchers (see [6, 7] for classic methods), a recent survey can be found in [8].

We consider the phase retrieval problem for discrete 1D real signals. In this case, it is well known that the mapping from signals to their Fourier transform magnitude is not one-to-one. In order to overcome this issue, researchers have tried various methods which can be broadly classified into two categories: (i) additional prior information (e.g., sparsity) [9–14] (ii) additional measurements [15–17].

In many signal processing applications, it is natural to define the Short-Time Fourier Transform (STFT) instead of the Fourier transform. In speech processing, the STFT magnitude is often transformed and the recovery of the transformed speech is essentially an STFT phase retrieval problem [18, 19]. In optics, this problem occurs in frequency resolved optical gating (FROG), which is a general method for measuring ultrashort laser pulses. Recovery of the pulse from its FROG trace involves STFT phase retrieval [20]. Ptychography [21], along with advances in detectors and computing have resulted in X-ray, optical and electron microscopy with increased spatial resolution without the need for advanced lenses. This procedure also involves STFT phase retrieval.

In this work, we explore the STFT phase retrieval problem. Our contribution is two-fold:

(i) Uniqueness guarantees: Researchers have previously explored deterministic conditions under which distinct signals cannot have the same STFT magnitude. However, either a lot prior information on the signal is assumed in order to provide the guarantees or the guarantees are very limited. For instance, the guarantees provided in [18] require the exact knowledge of a considerable portion of the underlying signal. In [22], guarantees are provided for the setup in which adjacent short-time sections differ in only one location.

These limitations are primarily due to a small number of adversarial signals which cannot be identified from their STFT magnitude. In this work, in contrast, we develop conditions under which the STFT magnitude is a unique signal representation *almost surely*. We show that almost all signals can be uniquely identified from their STFT magnitude if adjacent short-time sections overlap (Theorem 3.1).

(ii) Provable recovery algorithm: Researchers have developed efficient iterative algorithms to solve this problem based on theoretical grounds (Griffin-Lim [23], GESPAR [22]). While these algorithms work well in practice, they do not have provable recovery guarantees. Inspired by the success of convex relaxation-based techniques in solving certain problems provably [16, 24, 25], [26] proposed the use of a convex program to solve the STFT phase retrieval problem. In this work, we provide the *first* theoretical guarantees for the convex relaxation-based STFT phase retrieval algorithm (Theorem 4.1).

This paper is organized as follows. In Section 2, we mathematically formulate the STFT phase retrieval problem and establish the notation. Sections 3 and 4 contain the uniqueness guarantees and the recovery algorithm respectively. Numerical simulations are provided in Section 5.

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2. PROBLEM SETUP

Let $\mathbf{x} = (x[0], x[1], ..., x[N-1])$ be a discrete-time real signal of length N and $\mathbf{w} = (w[0], w[1], ..., w[W-1])$ be a window of length W. The STFT with respect to the window $\mathbf{w}$, denoted by $\mathbf{Y}_w$, can be defined as follows:

$$Y_w[m,k] = \sum_{n=0}^{N-1} x[n]w[mL-n]e^{-j2\pi kn/N} \tag{1}$$

for $0 \leq k \leq N-1$ and $0 \leq m \leq M-1$, where L is the separation in time between adjacent short-time sections and $M = \lfloor \frac{N+W-1}{L} \rfloor$ is the number of short-time sections considered. $\mathbf{Y}_w$ is an $N \times M$ matrix, the m^{th} column of which can be viewed as the N-DFT of the signal obtained by multiplying the signal $\mathbf{x}$ with the flipped and mL time-shifted window $\mathbf{w}$.

The STFT phase retrieval problem can be mathematically stated as

$$\begin{aligned} &\text{find} && \mathbf{x} && \text{(P)} \\ &\text{s.t.} && |Y_w[m,k]| = \left| \sum_{n=0}^{N-1} x[n]w[mL-n]e^{-j2\pi kn/N} \right| \\ &&& \text{for} \quad 0 \leq k \leq N-1 \quad \& \quad 0 \leq m \leq M-1. \end{aligned}$$

Trivial ambiguities: The Fourier phase retrieval problem has three trivial ambiguities: global sign, time-shift and time-reversal [9,14]. In other words, signals which differ from each other only by a global sign, time-shift and/or time-reversal cannot be distinguished from each other from their Fourier transform magnitude. In the STFT phase retrieval problem, the global sign of the signal cannot be recovered. However, time-shift and time-reversal ambiguities can be resolved for *some* choices of $\mathbf{w}$, W and L.

We define the following notations for convenience:

- $\mathbf{x}$ is nowhere-vanishing if $\{x[n] \neq 0 \colon n \in [0, N-1]\}$. $\mathbf{w}$ is nowhere-vanishing if $\{w[n] \neq 0 \colon n \in [0, W-1]\}$.
- $\tilde{\mathbf{w}}_m$ is the signal obtained by shifting the flipped window by mL time slots (it has non-zero entries in the region $[mL-W+1, mL]$).
- $\odot$ is the Hadamard product operator (entrywise multiplication of two same-length objects).
- t_m and T_m denote the locations of the first and the last non-zero entries of $\{\mathbf{x} \odot \tilde{\mathbf{w}}_m\}$ for $0 \leq m \leq M-1$.
- $\equiv$ implies equality up to a sign.

3. UNIQUE RECOVERY

In this section, we provide conditions under which (P) *almost always* has a unique solution. We use a technique commonly known as *dimension counting* [27]. Our arguments can be summarized as follows: the set of all signals $\mathbf{x}$ of length N can be mapped to $\mathbb{R}^N$, which is a vector space of dimension N. The signals in this set which cannot be uniquely represented by their STFT magnitude (the set of violations) can be viewed as solutions of a bilinear system of equations (Lemma 3.1). Using this property, we show that the set of violations is a manifold of dimension strictly less than N under mild conditions (Lemma 3.2). Since the set of violations, i.e., the set of signals which cannot be uniquely represented by their STFT magnitude, is *measure zero* with respect to the set of all signals, almost all signals can be uniquely represented by their STFT magnitude.

Theorem 3.1. *Almost all signals can be uniquely recovered (up to global sign) from their STFT magnitude if*

1. $L < W \leq N/2$
2. $\mathbf{w}$ *is nowhere-vanishing.*

Proof. The set of signals $\mathbf{x}$ which are not nowhere-vanishing is a manifold of dimension strictly less than N. We discard these signals (equivalent to classifying them as non-recoverable) and consider only nowhere-vanishing signals.

In Lemma 3.1, we characterize the set of nowhere-vanishing signals that cannot be uniquely identified by their STFT magnitude. Using the aforementioned characterization, we show in Lemma 3.2 that almost all nowhere-vanishing signals are such that for any $0 \leq m \leq M-1$, $\{\mathbf{x} \odot \tilde{\mathbf{w}}_m\}$ can be uniquely identified (up to a sign) from the STFT magnitude if $L < W \leq N/2$ and $\mathbf{w}$ is nowhere-vanishing. Union bounding over all m, we deduce that almost all nowhere-vanishing signals are such that $\{\mathbf{x} \odot \tilde{\mathbf{w}}_m\}$ can be uniquely identified up to a sign for all $0 \leq m \leq M-1$. Since for $L < W$, adjacent sections overlap, the entire signal can be uniquely identified up to a *global* sign. □

Lemma 3.1. *Consider two nowhere-vanishing signals* $\mathbf{x}^{(a)} \not\equiv \mathbf{x}^{(b)}$ *of length N which have the same STFT magnitude. For each m, there exists signals* $\mathbf{g}^{(m)}$ *and* $\mathbf{h}^{(m)}$ *of lengths l_{gm} and l_{hm} respectively such that*

- $\mathbf{x}^{(a)} \odot \tilde{\mathbf{w}}_m \equiv \mathbf{g}^{(m)} \star \mathbf{h}^{(m)}$, $\mathbf{x}^{(b)} \odot \tilde{\mathbf{w}}_m \equiv \mathbf{g}^{(m)} \star \tilde{\mathbf{h}}^{(m)}$
- $l_{gm} + l_{hm} - 1 = T_m - t_m + 1$
- $g^{(m)}[0] = 1$, $g^{(m)}[l_{gm}-1] \neq 0$, $h^{(m)}[0] \neq 0$, $h^{(m)}[l_{hm}-1] \neq 0$

where $\tilde{\mathbf{h}}$ is the flipped version of $\mathbf{h}$.

Proof. In [10] (Lemma 2.1), it is shown that if two $(\leq N)$-length signals have the same $2N$-DFT magnitude, there exists signals $\mathbf{g}$ and $\mathbf{h}$ of lengths l_g and l_h with the aforementioned properties. Since the m^{th} column of STFT magnitude corresponds to N-DFT magnitude of a $(\leq W)$-length signal (where $W \leq N/2$), we can apply Lemma 2.1 from [10] to each column of the STFT magnitude. □

Lemma 3.2. *Almost all nowhere-vanishing signals* $\mathbf{x} \in \mathbb{R}^N$ *are such that* $\{\mathbf{x} \odot \tilde{\mathbf{w}}_m\}$ *can be uniquely identified (up to a sign) by the STFT magnitude, for any* $0 \leq m \leq M-1$*, if* $L < W \leq N/2$.

Proof. Let us focus our attention on short-time sections m and $m+1$ for a given m. Since the m^{th} window starts at t_m and the $(m+1)^{th}$ window ends at T_{m+1}, the set of all signals $\{\mathbf{x} \odot \tilde{\mathbf{w}}_m, \mathbf{x} \odot \tilde{\mathbf{w}}_{m+1}\}$ can be mapped to a vector space of dimension $T_{m+1} - t_m + 1$. We will show that the set of signals $\{\mathbf{x} \odot \tilde{\mathbf{w}}_m, \mathbf{x} \odot \tilde{\mathbf{w}}_{m+1}\}$ which cannot be uniquely identified by the m^{th} and $(m+1)^{th}$ column of the STFT magnitude is a manifold of dimension at most $T_{m+1} - t_m$ if $L < W \leq N/2$.

Suppose $\{\mathbf{x}^{(a)} \odot \tilde{\mathbf{w}}_m, \mathbf{x}^{(a)} \odot \tilde{\mathbf{w}}_{m+1}\} \not\equiv \{\mathbf{x}^{(b)} \odot \tilde{\mathbf{w}}_m, \mathbf{x}^{(b)} \odot \tilde{\mathbf{w}}_{m+1}\}$ have the same $\{|Y_w[j,k]| : m \leq j \leq m+1 \;\&\; 0 \leq k \leq N-1\}$. There can be three possible cases:

- $\mathbf{x}^{(a)} \odot \tilde{\mathbf{w}}_m \not\equiv \mathbf{x}^{(b)} \odot \tilde{\mathbf{w}}_m$, $\mathbf{x}^{(a)} \odot \tilde{\mathbf{w}}_{m+1} \equiv \mathbf{x}^{(b)} \odot \tilde{\mathbf{w}}_{m+1}$
- $\mathbf{x}^{(a)} \odot \tilde{\mathbf{w}}_m \equiv \mathbf{x}^{(b)} \odot \tilde{\mathbf{w}}_m$, $\mathbf{x}^{(a)} \odot \tilde{\mathbf{w}}_{m+1} \not\equiv \mathbf{x}^{(b)} \odot \tilde{\mathbf{w}}_{m+1}$
- $\mathbf{x}^{(a)} \odot \tilde{\mathbf{w}}_m \not\equiv \mathbf{x}^{(b)} \odot \tilde{\mathbf{w}}_m$, $\mathbf{x}^{(a)} \odot \tilde{\mathbf{w}}_{m+1} \not\equiv \mathbf{x}^{(b)} \odot \tilde{\mathbf{w}}_{m+1}$.

We will provide the proof for the first case; the other two cases can be proved using the same arguments.

From Lemma 3.1, we know that there exists signals $\mathbf{g}$ and $\mathbf{h}$ such that

$$\mathbf{x}^{(a)} \odot \tilde{\mathbf{w}}_m \equiv \mathbf{g} \star \mathbf{h} \quad \& \quad \mathbf{x}^{(b)} \odot \tilde{\mathbf{w}}_m \equiv \mathbf{g} \star \tilde{\mathbf{h}}. \qquad (2)$$

Note that $l_{hm} + l_{gm} - 1 = T_m - t_m + 1$. Since we do not know the values of l_{hm} and l_{gm}, we will consider all possible values. For any l_{hm} and l_{gm}, the following statements hold.

The set of all signals $\mathbf{x} \odot \tilde{\mathbf{w}}_{m+1}$ can be mapped to a vector space of dimension $T_{m+1} - t_{m+1} + 1$. The choice of $\mathbf{x} \odot \tilde{\mathbf{w}}_{m+1}$ fixes $\mathbf{x} \odot \tilde{\mathbf{w}}_m$ in the region of overlap, hence $\mathbf{g} \star \mathbf{h}$ and $\mathbf{g} \star \tilde{\mathbf{h}}$ should satisfy the following equations:

$$\sum_{i=0}^{n} h[i] g[n - t_m - i] = w[mL - n] x[n] \qquad (3)$$

$$\sum_{i=0}^{n} h[l_{hm} - 1 - i] g[n - t_m - i] \equiv w[mL - n] x[n] \qquad (4)$$

for all $t_{m+1} \leq n \leq T_m$.

The system of equations in (3, 4) are bilinear in $\{\mathbf{g}, \mathbf{h}\}$. For such systems, it is well known that $\{\mathbf{g}, \mathbf{h}\}$ can be chosen from a manifold of dimension at most $g_{lm} + g_{hm} - 1 - r$, where r is the number of *independent* bilinear equations [10, 28]. There are at least $T_m - t_{m+1} + 2$ independent bilinear equations in (3, 4) if there is at least one overlapping location (which is true for $L < W$), which can be shown as follows:

The system of equations (3) decides the last $T_m - t_{m+1} + 1$ entries of $\{g[1], ..., g[l_{gm} - 1], h[1], ..., h[l_{hm} - 1]\}$ once the remaining entries are chosen. However, (4) at $n = T_m$ essentially is $h[0] = h[l_{hm} - 1]$, because of which $h[0]$ is also decided by the remaining entries. Hence, at least $T_m - t_{m+1} + 2$ entries of $\{\mathbf{g}, \mathbf{h}\}$ are decided. Hence $\{\mathbf{g}, \mathbf{h}\}$, or equivalently $\{\mathbf{x} \odot \tilde{\mathbf{w}}_m\}$, can be chosen from a manifold of dimension at most $(t_{m+1} - t_m - 1)$.

Note that in (4), *equivalent* sign is used as the equality is only up to a sign (the argument holds for both possible signs). For each of the three aforementioned cases, the set is a manifold of dimension at most $(t_{m+1} - t_m - 1)$.

Using a union bound, we deduce that the set of all signals $\{\mathbf{x} \odot \tilde{\mathbf{w}}_m, \mathbf{x} \odot \tilde{\mathbf{w}}_{m+1}\}$ which cannot be uniquely identified from the m^{th} and $(m+1)^{th}$ column of the STFT magnitude is a manifold of dimension at most $(T_{m+1} - t_{m+1} + 1) + (t_{m+1} - t_m - 1) = (T_{m+1} - t_m)$. Since the entries of $\mathbf{x}$ which do not belong to the short-time sections m and $m+1$ can be chosen from a vector space of dimension $N - (T_{m+1} - t_m)$, the set of all signals $\mathbf{x}$ for which $\{\mathbf{x} \odot \tilde{\mathbf{w}}_m\}$ cannot be uniquely identified by the STFT magnitude is a manifold of dimension at most $N - 1$ for any $0 \leq m \leq M - 1$. □

4. RECOVERY ALGORITHM

The STFT phase retrieval problem (P) is a quadratically-constrained problem. A technique, popularly known as *lifting*, has enjoyed success in solving some quadratically-constrained problems (for example, see [15, 16]). The steps can be summarized as follows: (i) embed the problem in a higher dimensional space using the transformation $\mathbf{X} = \mathbf{x}\mathbf{x}^T$, a process which converts the problem of recovering a signal with quadratic constraints into a problem of recovering a rank-one matrix with affine constraints (ii) relax the rank-one constraint to obtain a convex program.

A convex program (Algorithm 1) to solve the STFT phase retrieval problem was proposed in [26]. If the solution to the convex program is a unique rank-one matrix, then it is also the unique solution to the quadratically-constrained problem. While the solution to the convex program need not be rank one in general, many recent results in the compressed sensing [24] and matrix completion [25] community suggest that one can provide conditions which ensure that the convex program has a unique rank one solution. In this section, we provide conditions on $\mathbf{w}$, W and L which ensure that the convex program always has a unique rank one solution.

Theorem 4.1. *Algorithm 1 uniquely recovers (up to a global sign) a nowhere-vanishing signal* $\mathbf{x}$ *from its STFT magnitude if*

1. $L = 1$, $2 \leq W \leq N/2$
2. $w[0]w[1] \neq 0$.

Proof. For all $0 \leq m \leq M-1$, we can say the following:

$$\sum_{k=0}^{N-1} |Y_w[m,k]|^2 = trace\Big(\sum_{k=0}^{N-1} (\mathbf{f}_k \mathbf{f}_k^T)\, (\mathbf{X} \odot (\tilde{\mathbf{w}}_m \tilde{\mathbf{w}}_m^T)\,) \Big)$$

Algorithm 1 STFT Phase Retrieval Algorithm

Input: STFT magnitude measurements $\mathbf{Y}$, $\mathbf{w}$, W, L
Output: Signal $\mathbf{x}_\star$

- Solve for $\mathbf{X}_\star$

$$\begin{aligned} &\text{minimize} \quad trace(\mathbf{X}) && \text{(R)} \\ &\text{subject to} \quad |Y_w[m,k]|^2 = trace(\ \mathbf{f}_k \mathbf{f}_k^T\ (\mathbf{X} \odot (\tilde{\mathbf{w}}_m \tilde{\mathbf{w}}_m^T)\ ) \\ &\qquad\qquad \text{for}\ \ 0 \le m \le M-1\ \ \&\ \ 0 \le k \le N-1 \\ &\qquad\qquad \mathbf{X} \succcurlyeq 0 \end{aligned}$$

where $\mathbf{f}_k$ is the k^{th} column of the N-DFT matrix.

- Return $\mathbf{x}_\star$, where $\mathbf{x}_\star \mathbf{x}_\star^T$ is the best rank-one approximation of $\mathbf{X}_\star$

$$= trace(\ (\mathbf{X} \odot (\tilde{\mathbf{w}}_m \tilde{\mathbf{w}}_m^T)\ ) = \sum_{n=t_m}^{T_m} X[n,n] w^2[T_m - n]. \quad (5)$$

Using Parseval's theorem, (5) can be rewritten as

$$\sum_{n=t_m}^{T_m} X[n,n] w^2[T_m - n] = \sum_{n=t_m}^{T_m} x^2[n] w^2[T_m - n]. \quad (6)$$

Similarly, considering the sum $\sum_{k=0}^{N-1} e^{j2\pi k/N} |Y_w[m,k]|^2$, we can say the following:

$$\begin{aligned} &\sum_{n=t_m}^{T_m - 1} X[n, n+1] w[T_m - n] w[T_m - n - 1] \\ &= \sum_{n=t_m}^{T_m - 1} x[n] x[n+1] w[T_m - n] w[T_m - n - 1] \quad (7) \end{aligned}$$

For $m = 1$, (6) and (7) correspond to

$$w^2[0] X[0,0] = w^2[0] x^2[0]$$

which fixes $X[0,0]$ to $x^2[0]$ if $w[0] \neq 0$. For $m = 2$, (6) and (7) result in

$$\begin{aligned} w^2[0] X[1,1] + w^2[1] X[0,0] &= w^2[0] x^2[1] + w^2[1] x^2[0] \\ w[0] w[1] X[0,1] &= w[0] x[1] w[1] x[0]. \quad (8) \end{aligned}$$

If $X[0,0]$ is equal to $x^2[0]$, $W > 1$ and $w[0]w[1] \neq 0$, then $X[1,1]$ and $X[0,1]$ equal $x^2[1]$ and $x[0]x[1]$.

Applying this argument incrementally, (6) and (7) for measurement m, with the help of the entries fixed by previous measurements, sets $X[n-1, n-1]$ and $X[n-2, n-1]$ to $x^2[n-1]$ and $x[n-2]x[n-1]$ respectively, if $w[0]w[1] \neq 0$.

Hence, the diagonal and the first off-diagonal entries of $\mathbf{X}$ are fixed by the STFT magnitude measurements. If the diagonal and the first off-diagonal entries of a matrix are sampled from a rank-one matrix, there is precisely one positive semidefinite completion of the matrix and that is the rank-one completion [30].

In particular, since $\mathbf{X} \succcurlyeq 0$, (R) has only one feasible point, given by $\{\mathbf{X}, X[i,j] = x[i]x[j]\text{: } 0 \le i, j \le N-1\}$. Hence, the solution to (R) is a rank-one matrix $\mathbf{x}\mathbf{x}^T$, from which the true signal $\mathbf{x}$ can be recovered (up to a global sign) by a simple decomposition if $\mathbf{x}$ is nowhere-vanishing. □

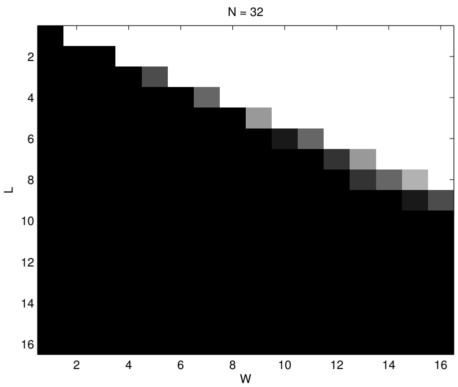

Fig. 1. Probability of successful recovery of Algorithm 1 for $N = 32$ for various choices of L and W (white region: success with probability 1).

5. NUMERICAL SIMULATIONS

In this section, we evaluate the probability of successful signal recovery of Algorithm 1. For $N = 32$, we vary both L and W between 1 and $N/2$. For each L and W, we performed 100 simulations by randomly choosing a nowhere-vanishing signal $\mathbf{x}$ from an i.i.d Gaussian distribution and a window $\mathbf{w}$ with unit entries, and recorded the number of times the algorithm successfully recovered the underlying signal exactly (Fig. 1).

We observed that Algorithm 1 successfully recovers the underlying signal with very high probability if $2L \le W \le N/2$. The $\{L = N/4, W = N/2\}$ case uses only *six* measurements and Algorithm 1 managed to recover the underlying signal with very high probability, which, given the limited success of semidefinite relaxation-based algorithms in the Fourier phase retrieval setup [29], we found surprising.

6. FUTURE WORK

Simulations strongly suggest that Theorem 4.1 can be generalized to $2L \le W \le N/2$. We leave this for future work. Also, there is a sharp phase transition at $2L = W$ (Fig. 1), i.e., recovery is successful with very high probability if $2L \le W$ and fails with very high probability if $2L > W$. A theoretical analysis of this phase transition would be a very interesting direction of future study.

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